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Generalized multi-mode bosonic realization of the $SU(1, 1)$ algebra and its corresponding squeezing operator

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Abstract
By constructing a generalized multi-partite entangled state representation and introducing the ket-bra integral in this representation, we find a new set of generalized bosonic realization of the generators of the $SU(1, 1)$ algebra, which can compose a generalized multi-mode squeezing operator. This operator squeezes the multi-partite entangled state in a natural way. Then the corresponding multi-mode squeezed vacuum states $|r\rangle$ is obtained. Based on this, the variances of the n-mode quadratures and the higher-order squeezing in $|r\rangle$ are evaluated. In addition, we examine the violation of the Bell inequality for $|r\rangle$ by using the formalism of Wigner representation.

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1. Introduction

It has long been known that the bosonic realizations of the $SU(1, 1)$ algebra have applications in many branches of physics and group theory [1–4]. The generators of the $SU(1, 1)$ algebra are given by K_0 and $K_\pm$ with the commutative relations

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0. \tag{1}$$

The $SU(1, 1)$ Casimir operator is

$$C = K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+). \tag{2}$$

In particular, the $SU(1, 1)$ Lie algebra was widely used in quantum optics [5–7]. For example, the $SU(1, 1)$ coherent states, defined by Perelomov [8], have previously been discussed in connection with squeezed states of a single-mode field and are a special case of the two-photon coherent states of Yuen [9], namely the squeezed vacuum state [10]. The single-mode bosonic realization of $SU(1, 1)$ is

$$K_+ \to \frac{a^{\dagger 2}}{2}, \qquad K_- \to \frac{a^2}{2}, \qquad K_0 \to \frac{1}{4}(2a^\dagger a + 1), \tag{3}$$

where $a^\dagger$ and a are the bosonic creation and annihilation operators, respectively; in this case the Casimir operator is $C = -3/16$. The corresponding squeezing operator [10],

$$\begin{aligned} S_1(r) &= \exp\left[\frac{r}{2}(a^{\dagger 2} - a^2)\right] \\ &= \exp\left(\frac{a^{\dagger 2}}{2}\tanh r\right)\exp\left[\left(a^\dagger a + \frac{1}{2}\right)\ln \operatorname{sech} r\right]\exp\left(-\frac{a^2}{2}\tanh r\right), \end{aligned} \tag{4}$$

acting on the vacuum state $|0\rangle$ leads to the single-mode squeezed vacuum state

$$S_1(r)|0\rangle = (\operatorname{sech} r)^{1/2}\exp\left(\frac{a^{\dagger 2}}{2}\tanh r\right)|0\rangle, \tag{5}$$

where r is a real squeezing parameter. The two-mode bosonic realization for $SU(1, 1)$ is

$$K_+ \to a_1^\dagger a_2^\dagger, \qquad K_- \to a_1 a_2, \qquad K_0 \to \tfrac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right), \tag{6}$$

with the Casimir operator $C = \left[\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)^2 - 1\right]/4$, the two-mode squeezing operator [11]

$$\begin{aligned} S_2(r) &= \exp\left[r\left(a_1^\dagger a_2^\dagger - a_1 a_2\right)\right] \\ &= \exp\left(a_1^\dagger a_2^\dagger \tanh r\right)\exp\left[\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)\ln \operatorname{sech} r\right]\exp(-a_1 a_2 \tanh r) \end{aligned} \tag{7}$$

produces the two-mode squeezed vacuum state

$$S_2(r)|00\rangle = \operatorname{sech} r \exp\left(a_1^\dagger a_2^\dagger \tanh r\right)|00\rangle. \tag{8}$$

Similarly, for the n-mode case, the squeezing operator is given by [12, 13]

$$S_n(r) = \exp[r(W_+ - W_-)], \tag{9}$$

where

$$W_+ = W_-^\dagger = \frac{2-n}{2n}\sum_{j=1}^{n} a_j^{\dagger 2} + \frac{2}{n}\sum_{j>k=1}^{n} a_j^\dagger a_k^\dagger, \tag{10}$$

satisfying a closed $SU(1, 1)$ Lie algebra,

$$[W_-, W_+] = 2W_0, \qquad [W_0, W_+] = W_+, \qquad [W_0, W_-] = -W_-, \tag{11}$$

with

$$W_0 = \frac{1}{2}\sum_{j=1}^{n} a_j^\dagger a_j + \frac{n}{4}. \tag{12}$$

These squeezing operators may be called $SU(1, 1)$ operators and the corresponding squeezed vacuum states are named $SU(1, 1)$ coherent states as well. Two interesting questions naturally arise. Are there more generalized bosonic operator realization of $SU(1, 1)$ generators for generalized squeezing operators? If yes, how do we find them? To answer the second question we recall the relation of the two-mode squeezing operator and the bipartite entangled state representation, i.e. in [14] we have proved

$$S_2(r) = \int \frac{d^2\eta}{\mu\pi}\left|\frac{\eta}{\mu}\right\rangle\langle\eta|, \qquad \eta = \eta_1 + i\eta_2, \qquad d^2\eta = d\eta_1\, d\eta_2, \tag{13}$$

where $\mu = \mathrm{e}^r$, and

$$|\eta\rangle = \exp\left(-\tfrac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right)|00\rangle \tag{14}$$

is the common eigenvectors of the relative position $X_1 - X_2$ and the total momentum $P_1 + P_2$ of two particles, i.e.

$$(X_1 - X_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle, \qquad (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle. \tag{15}$$

It is Einstein–Podolsky–Rosen who first used $[X_1 - X_2, P_1 + P_2] = 0$ to introduce the concept of quantum entanglement [15]. Directly performing the integration over the ket-bra $\left|\frac{\eta}{\mu}\right\rangle\langle\eta|$ by virtue of the technique of integration within an ordered product (IWOP) of operators [16] leads to the right-hand side of equation (7), which shows that the two-mode squeezing operator $S_2(r)$ has a natural representation in the entangled state $|\eta\rangle$. Equation (13) shows that by constructing generalized multi-partite entangled state representation we may find the generalized bosonic operator realization of $SU(1,1)$ generators.

The organization of this paper is as follows. In section 2, we briefly review multi-partite entangled state representations $|\chi,\vec{\rho}\rangle$ and $|\rho,\vec{\chi}\rangle$. In section 3, by constructing a ket-bra integral in the $|\chi,\vec{\rho}\rangle$ representation and using the IWOP technique, we derive the generalized n-mode squeezing operator $U_n(r)$, which involves bosonic realization of the generalized $SU(1,1)$ generators. We then discuss the transformation properties of $a_i^\dagger$ and a_i under the operation of $U_n(r)$ and give the interaction Hamiltonian generating such an $U_n(r)$. In section 4, we evaluate the variances of the n-mode quadratures and higher order squeezing for the generalized n-mode squeezed vacuum state $U_n(r)|\vec{0}\rangle \equiv |r\rangle$. Section 5 is devoted to deriving the expression of the Wigner function of $|r\rangle$ and examining its violation of the Bell inequality by using the formalism of Wigner representation in phase space.

2. Multi-partite entangled state representations

We begin with briefly introducing the multi-partite entangled state representations and listing some of their properties. For an n-partite system, let

$$X = \sum_{j=1}^{n} \varepsilon_j X_j \tag{16}$$

denote the center-of-mass coordinate, where $\varepsilon_j = m_j/M$ $\left(M = \sum_{j=1}^n m_j\right)$ is the ratio of each particle's mass to the total mass, $\sum_{j=1}^n \varepsilon_j = 1$. X is permutable with the mass-weighted relative momentum $\frac{P_1}{\varepsilon_1} - \frac{P_j}{\varepsilon_j}$ $(j = 2, 3, \ldots, n)$, i.e.

$$\left[X, \frac{P_1}{\varepsilon_1} - \frac{P_j}{\varepsilon_j}\right] = 0 \tag{17}$$

and

$$\left[\frac{P_1}{\varepsilon_1} - \frac{P_j}{\varepsilon_j}, \frac{P_1}{\varepsilon_1} - \frac{P_k}{\varepsilon_k}\right] = 0 \qquad (j \neq k), \tag{18}$$

where P_j is the momentum of particle j. In [17], we have derived the common eigenvector of X and $\frac{P_1}{\varepsilon_1} - \frac{P_j}{\varepsilon_j}$ $(j = 2, 3, \ldots, n)$, in the n-mode Fock space expressed as

$$|\chi,\vec{\rho}\rangle = D_1 \exp\left\{\frac{\sqrt{2}\chi}{\lambda}\sum_{j=1}^{n}\varepsilon_j a_j^\dagger + \frac{\mathrm{i}\sqrt{2}}{\lambda}\sum_{j=2}^{n}\varepsilon_j^2\rho_j\left(\sum_{k=1}^{n}\varepsilon_k a_k^\dagger - \frac{\lambda}{\varepsilon_j}a_j^\dagger\right)\right.$$
$$\left. + \sum_{j=1}^{n}\left(\frac{1}{2} - \frac{\varepsilon_j^2}{\lambda}\right)a_j^{\dagger 2} - \frac{2}{\lambda}\sum_{k>j=1}^{n}\varepsilon_k\varepsilon_j a_k^\dagger a_j^\dagger\right\}|\vec{0}\rangle, \tag{19}$$

where $\vec{\rho} \equiv \rho_2, \rho_3, \ldots, \rho_n$, $\lambda \equiv \sum_{j=1}^{n} \varepsilon_j^2$ and $|\vec{0}\rangle$ is the n-mode vacuum state

$$D_1 \equiv \pi^{-n/4}\sqrt{\frac{\prod_{j=1}^{n}\varepsilon_j}{\lambda}}\exp\left[-\frac{1}{2\lambda}\left(\chi^2 + \sum_{j=2}^{n}\varepsilon_j^2\rho_j^2\left(\lambda-\varepsilon_j^2\right) - 2\sum_{k>j=2}^{n}\varepsilon_k^2\varepsilon_j^2\rho_k\rho_j\right)\right]. \tag{20}$$

In fact, noting

$$X_j = \frac{1}{\sqrt{2}}\left(a_j + a_j^\dagger\right), \qquad P_j = \frac{1}{\mathrm{i}\sqrt{2}}\left(a_j - a_j^\dagger\right), \tag{21}$$

one can check that $|\chi, \vec{\rho}\rangle$ obeys the eigenvector equations

$$X|\chi, \vec{\rho}\rangle = \chi|\chi, \vec{\rho}\rangle \tag{22}$$

and

$$\left(\frac{P_1}{\varepsilon_1} - \frac{P_j}{\varepsilon_j}\right)|\chi, \vec{\rho}\rangle = \rho_j\,|\chi, \vec{\rho}\rangle, \qquad j = 2, 3, \ldots, n. \tag{23}$$

Using $|\vec{0}\rangle\langle\vec{0}| =\colon \exp\left(-\sum_{j=1}^{n} a_j^\dagger a_j\right)\colon$ ($\colon\,\colon$ denotes normal ordering), and the integration with an ordered product of operators (IWOP) technique [16], we can prove that $|\chi, \vec{\rho}\rangle$ really spans an orthogonal and complete set, i.e.

$$\langle\chi', \vec{\rho}'|\chi, \vec{\rho}\rangle = \delta(\chi' - \chi)\delta(\rho_2' - \rho_2)\cdots\delta(\rho_n' - \rho_n) \tag{24}$$

and

$$\int_{-\infty}^{+\infty} \mathrm{d}\chi\,\mathrm{d}\vec{\rho}\,|\chi, \vec{\rho}\rangle\,\langle\chi, \vec{\rho}| = 1, \tag{25}$$

where $\mathrm{d}\vec{\rho} \equiv \mathrm{d}\rho_2\,\mathrm{d}\rho_3\cdots\mathrm{d}\rho_n$.

We may also introduce the common eigenvector $|\rho, \vec{\chi}\rangle$ of the operator $P = \sum_{j=1}^{n}\varepsilon_j P_j$ and $\left(\frac{X_1}{\varepsilon_1} - \frac{X_j}{\varepsilon_j}\right)$,

$$P|\rho, \vec{\chi}\rangle = \rho|\rho, \vec{\chi}\rangle \tag{26}$$

and

$$\left(\frac{X_1}{\varepsilon_1} - \frac{X_j}{\varepsilon_j}\right)|\rho, \vec{\chi}\rangle = \chi_j|\rho, \vec{\chi}\rangle, \qquad j = 2, 3, \ldots, n, \tag{27}$$

with $\vec{\chi} \equiv \chi_2, \chi_3, \ldots, \chi_n$ and

$$|\rho, \vec{\chi}\rangle = D_2\exp\left\{\frac{\mathrm{i}\sqrt{2}\rho}{\lambda}\sum_{j=1}^{n}\varepsilon_j a_j^\dagger + \frac{\sqrt{2}}{\lambda}\sum_{j=2}^{n}\varepsilon_j^2\chi_j\left(\sum_{k=1}^{n}\varepsilon_k a_k^\dagger - \frac{\lambda}{\varepsilon_j}a_j^\dagger\right)\right.$$
$$\left. - \sum_{j=1}^{n}\left(\frac{1}{2} - \frac{\varepsilon_j^2}{\lambda}\right)a_j^{\dagger 2} + \frac{2}{\lambda}\sum_{k>j=1}^{n}\varepsilon_k\varepsilon_j a_k^\dagger a_j^\dagger\right\}|\vec{0}\rangle, \tag{28}$$

where

$$D_2 \equiv \pi^{-n/4}\sqrt{\frac{\prod_{j=1}^{n}\varepsilon_j}{\lambda}}\exp\left[-\frac{1}{2\lambda}\left(\rho^2 + \sum_{j=2}^{n}\varepsilon_j^2\chi_j^2\left(\lambda-\varepsilon_j^2\right) - 2\sum_{k>j=2}^{n}\varepsilon_k^2\varepsilon_j^2\chi_k\chi_j\right)\right], \tag{29}$$

which obeys the completeness relation $\int_{-\infty}^{+\infty}\mathrm{d}\rho\,\mathrm{d}\vec{\chi}\,|\rho, \vec{\chi}\rangle\langle\rho, \vec{\chi}| = 1$ with $\mathrm{d}\vec{\chi} \equiv \mathrm{d}\chi_2\,\mathrm{d}\chi_3\cdots\mathrm{d}\chi_n$ as well.

3. Generalized multi-mode $SU(1,1)$ generators and squeezing operator

As we have seen, in the bipartite case the ket-bra integration $\int \frac{d^2\eta}{\mu\pi}\left|\frac{\eta}{\mu}\right\rangle\langle\eta|$ can produce two-mode $SU(1,1)$ generators and form the two-mode squeezing operator (see equations (6) and (7)), so we construct the following ket-bra integral in the $|\chi,\vec{\rho}\rangle$ representation:

$$U_n(\mu) = \mu^{n/2}\int_{-\infty}^{+\infty} d\chi\, d\vec{\rho}|\mu\chi,\mu\vec{\rho}\rangle\langle\chi,\vec{\rho}|. \tag{30}$$

3.1. Performing the above integration to find $SU(1,1)$ generators

Substituting equation (19) into equation (30) and considering $|\vec{0}\rangle\langle\vec{0}| =: \exp\left(-\sum_{j=1}^{n} a_j^\dagger a_j\right):$, after some lengthy but straightforward calculation we arrive at the following result by virtue of the IWOP technique [16]:

$$\begin{aligned}
U_n(\mu) = {} & \frac{\prod_{j=1}^{n}\varepsilon_j}{\lambda}\left(\frac{\mu}{\pi}\right)^{n/2}\int_{-\infty}^{+\infty} d\chi\, d\vec{\rho} : \exp\left\{-\frac{\mu^2+1}{2\lambda}\left[\chi^2 + \sum_{j=2}^{n}\varepsilon_j^2\rho_j^2(\lambda-\varepsilon_j^2)\right.\right. \\
& \left. -2\sum_{k>j=2}^{n}\varepsilon_k^2\varepsilon_j^2\rho_j\rho_k\right] + \frac{\sqrt{2}}{\lambda}\sum_{j=1}^{n}\varepsilon_j\left(\mu a_j^\dagger + a_j\right)\chi + \frac{i\sqrt{2}}{\lambda}\sum_{j=2}^{n}\varepsilon_j^2\rho_j \\
& \times\left[\sum_{k=1}^{n}\varepsilon_k\left(\mu a_k^\dagger - a_k\right) - \frac{\lambda}{\varepsilon_j}\left(\mu a_j^\dagger - a_j\right)\right] + \sum_{j=1}^{n}\left(\frac{1}{2}-\frac{\varepsilon_j^2}{\lambda}\right)a_j^{\dagger 2} \\
& \left. -\frac{2}{\lambda}\sum_{k>j=1}^{n}\varepsilon_k\varepsilon_j a_k^\dagger a_j^\dagger + \sum_{j=1}^{n}\left(\frac{1}{2}-\frac{\varepsilon_j^2}{\lambda}\right)a_j^2 - \frac{2}{\lambda}\sum_{k>j=1}^{n}\varepsilon_k\varepsilon_j a_k a_j - \sum_{j=1}^{n} a_j^\dagger a_j\right\} : \\
= {} & \operatorname{sech}^{n/2} r : \exp\left\{\left[-\frac{1}{2}\sum_{j=1}^{n}\left(1-\frac{2\varepsilon_j^2}{\lambda}\right)a_j^{\dagger 2} + \frac{2}{\lambda}\sum_{k>j=1}^{n}\varepsilon_k\varepsilon_j a_k^\dagger a_j^\dagger\right]\tanh r\right. \\
& \left. + (\operatorname{sech} r - 1)\sum_{j=1}^{n} a_j^\dagger a_j + \left[\frac{1}{2}\sum_{j=1}^{n}\left(1-\frac{2\varepsilon_j^2}{\lambda}\right)a_j^2 - \frac{2}{\lambda}\sum_{k>j=1}^{n}\varepsilon_k\varepsilon_j a_k a_j\right]\tanh r\right\} : \\
\equiv {} & U_n(r),
\end{aligned} \tag{31}$$

where $\mu \equiv e^r$, $\operatorname{sech} r = 2\mu/(\mu^2+1)$, $\tanh r = (\mu^2-1)/(\mu^2+1)$ and we have used the integral formula

$$\int dy \exp(-fy^2 + gy) = \left(\frac{\pi}{f}\right)^{1/2}\exp\left(\frac{g^2}{4f}\right), \qquad (f > 0). \tag{32}$$

Using the formula $\exp[(e^{\upsilon}-1)a^\dagger a] := e^{\upsilon a^\dagger a}$ and letting

$$V_+ = V_-^\dagger = \frac{1}{2}\sum_{j=1}^{n}\left(\frac{2\varepsilon_j^2}{\lambda}-1\right)a_j^{\dagger 2} + \frac{2}{\lambda}\sum_{k>j=1}^{n}\varepsilon_k\varepsilon_j a_k^\dagger a_j^\dagger, \tag{33}$$

we can simplify equation (31) as

$$\begin{aligned}
U_n(r) &= \exp(V_+\tanh r)\exp(2W_0\ln\operatorname{sech} r)\exp(-V_-\tanh r) \\
&= \exp[r(V_+ - V_-)],
\end{aligned} \tag{34}$$

where V_+ and V_- are just the generalized multi-mode bosonic realization of the $SU(1,1)$ algebra,

$$[V_-, V_+] = 2W_0, \qquad [W_0, V_+] = V_+, \qquad [W_0, V_-] = -V_-; \tag{35}$$

here W_0 has been defined in equation (12). In particular, when $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = \frac{1}{n}$ and $\lambda = \frac{1}{n}$,

$$V_+ = W_+ \tag{36}$$

and

$$U_n(r) \rightarrow S_n(r). \tag{37}$$

Therefore, we can call $U_n(r)$ the generalized multi-mode $SU(1,1)$ squeezing operator. From equations (24) and (30), we see

$$U_n(\mu)|\chi, \vec{\rho}\rangle = \mu^{n/2}|\mu\chi, \mu\vec{\rho}\rangle; \tag{38}$$

thus $|\chi, \vec{\rho}\rangle$ is a natural representation for the squeezing operator $U_n(r)$.

3.2. The behavior under $U_n(r)$ transformation

In order to simplify the form of V_+ in equation (33), we introduce the symmetric matrix G

$$G = \begin{pmatrix} \frac{2\varepsilon_1^2}{\lambda} - 1 & \frac{2}{\lambda}\varepsilon_1\varepsilon_2 & \cdots & \frac{2}{\lambda}\varepsilon_1\varepsilon_n \\ \frac{2}{\lambda}\varepsilon_2\varepsilon_1 & \frac{2\varepsilon_2^2}{\lambda} - 1 & \cdots & \frac{2}{\lambda}\varepsilon_2\varepsilon_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{\lambda}\varepsilon_n\varepsilon_1 & \frac{2}{\lambda}\varepsilon_n\varepsilon_2 & \cdots & \frac{2\varepsilon_n^2}{\lambda} - 1 \end{pmatrix}_{n\times n}, \tag{39}$$

which obeys

$$G^2 = I, \qquad \sum_{j,k=1}^{n} G_{jk} = \frac{2}{\lambda} - n \tag{40}$$

and

$$\mathrm{e}^{rG} = I\cosh r + G\sinh r. \tag{41}$$

With the help of G, V_+ is re-expressed as

$$V_+ = \tfrac{1}{2}a_j^\dagger G_{jk}a_k^\dagger \tag{42}$$

and

$$U_n(r) = \exp\left[\frac{r}{2}\left(a_j^\dagger G_{jk}a_k^\dagger - a_jG_{jk}a_k\right)\right]; \tag{43}$$

here and henceforth the repeated indices imply the Einstein summation notation.

Therefore, due to equations (43) and (41), using the Baker–Hausdorff formula

$$\mathrm{e}^{A}B\,\mathrm{e}^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots, \tag{44}$$

and $\left[a_j, a_k^\dagger\right] = \delta_{jk}$, we have

$$U_n^{-1}(r)a_jU_n(r) = a_j\cosh r + a_k^\dagger G_{kj}\sinh r \tag{45}$$

and

$$U_n^{-1}(r)a_j^\dagger U_n(r) = a_j^\dagger \cosh r + a_k G_{kj} \sinh r. \tag{46}$$

It then follows

$$U_n^{-1}(r)X_j U_n(r) = X_j \cosh r + X_k G_{kj} \sinh r = X_k (\mathrm{e}^{rG})_{kj}, \tag{47}$$

and

$$U_n^{-1}(r)P_j U_n(r) = P_j \cosh r - P_k G_{kj} \sinh r = P_k (\mathrm{e}^{-rG})_{kj}. \tag{48}$$

We can further check

$$U_n^{-1}(r)XU_n(r) = \varepsilon_j X_j \cosh r + \varepsilon_j X_k G_{kj} \sinh r, \tag{49}$$

where

$$\begin{aligned}
\varepsilon_j X_k G_{kj} &= \varepsilon_j G_{jk} X_k \\
&= (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \begin{pmatrix} \frac{2\varepsilon_1^2}{\lambda} - 1 & \frac{2}{\lambda}\varepsilon_1\varepsilon_2 & \cdots & \frac{2}{\lambda}\varepsilon_1\varepsilon_n \\ \frac{2}{\lambda}\varepsilon_2\varepsilon_1 & \frac{2\varepsilon_2^2}{\lambda} - 1 & \cdots & \frac{2}{\lambda}\varepsilon_2\varepsilon_n \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{\lambda}\varepsilon_n\varepsilon_1 & \frac{2}{\lambda}\varepsilon_n\varepsilon_2 & \cdots & \frac{2\varepsilon_n^2}{\lambda} - 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \\
&= \varepsilon_1 \left[\left(\frac{2\varepsilon_1^2}{\lambda} - 1 \right) X_1 + \frac{2}{\lambda}\varepsilon_1\varepsilon_2 X_2 + \cdots + \frac{2}{\lambda}\varepsilon_1\varepsilon_n X_n \right] \\
&\quad + \varepsilon_2 \left[\frac{2}{\lambda}\varepsilon_2\varepsilon_1 X_1 + \left(\frac{2\varepsilon_2^2}{\lambda} - 1 \right) X_2 + \cdots + \frac{2}{\lambda}\varepsilon_2\varepsilon_n X_n \right] \\
&\quad + \cdots + \varepsilon_n \left[\frac{2}{\lambda}\varepsilon_n\varepsilon_1 X_1 + \frac{2}{\lambda}\varepsilon_n\varepsilon_2 X_2 + \cdots + \left(\frac{2\varepsilon_n^2}{\lambda} - 1 \right) X_n \right] \\
&= (\varepsilon_1 X_1 + \varepsilon_2 X_2 + \cdots + \varepsilon_n X_n) \left(\frac{2\varepsilon_1^2}{\lambda} - 1 + \frac{2\varepsilon_2^2}{\lambda} + \cdots + \frac{2\varepsilon_n^2}{\lambda} \right) \\
&= X;
\end{aligned} \tag{50}$$

in the last step we have used $\lambda \equiv \sum_{j=1}^n \varepsilon_j^2$. Thus $U_n^{-1}(r)XU_n(r) = \mu X$, as expected. In fact, considering equations (22), (24) and (25), we can prove that

$$\begin{aligned}
U_n^{-1}(r)XU_n(r) &= \mu^n \int_{-\infty}^{+\infty} \mathrm{d}\chi\, \mathrm{d}\vec{\rho} |\chi, \vec{\rho}\rangle\langle\mu\chi, \mu\vec{\rho}| X \int_{-\infty}^{+\infty} \mathrm{d}\chi'\, \mathrm{d}\vec{\rho}' |\mu\chi', \mu\vec{\rho}'\rangle\langle\chi', \vec{\rho}'| \\
&= \mu^n \mu\chi \int_{-\infty}^{+\infty} \mathrm{d}\chi\, \mathrm{d}\vec{\rho} |\chi, \vec{\rho}\rangle\langle\mu\chi, \mu\vec{\rho}| \int_{-\infty}^{+\infty} \mathrm{d}\chi'\, \mathrm{d}\vec{\rho}' |\mu\chi', \mu\vec{\rho}'\rangle\langle\chi', \vec{\rho}'| \\
&= \mu \int_{-\infty}^{+\infty} \mathrm{d}\chi\, \mathrm{d}\vec{\rho} |\chi, \vec{\rho}\rangle\langle\chi, \vec{\rho}|\chi \\
&= \mu X.
\end{aligned} \tag{51}$$

3.3. *The interaction Hamiltonian*

Next, we seek the interaction Hamiltonian which can generate such a $U_n(r)$. For this purpose, we differentiate the two sides of equation (43) with respect to t and obtain

$$\mathrm{i}\frac{\partial U_n(r)}{\partial t} = \frac{\mathrm{i}}{2}\left(a_j^\dagger G_{jk} a_k^\dagger - a_j G_{jk} a_k\right)\frac{\partial r}{\partial t} U_n(r), \tag{52}$$

with $U_n(r)|_{t=0} = 1$, so the interaction Hamiltonian is

$$H_I(t) = \frac{\mathrm{i}}{2}\left[a_j^\dagger G_{jk} a_k^\dagger - a_j G_{jk} a_k\right]\frac{\partial r}{\partial t}. \tag{53}$$

Such a dynamic process may happen in some $2n$-wave mixing processes; for the four-wave mixing we refer to the pioneering paper of Yuen and Shapiro [18].

4. Generalized multi-mode $SU(1,1)$ squeezed vacuum states

Acting the operator $U_n(r)$ in equation (31) on the n-mode vacuum state $|\vec{0}\rangle$ and using $:F\left(a_i^\dagger, a_j\right):|\vec{0}\rangle = F\left(a_i^\dagger, 0\right)|\vec{0}\rangle$, we obtain the n-mode squeezed vacuum state

$$|r\rangle \equiv U_n(r)|\vec{0}\rangle = \mathrm{sech}^{n/2}\, r \exp(V_+ \tanh r)|\vec{0}\rangle, \tag{54}$$

where V_+ has been defined in equation (33). Now we evaluate the variances of two n-mode quadratures which are defined as [10]

$$X_0 = \frac{1}{\sqrt{2n}}\sum_{j=1}^{n} X_j, \qquad P_0 = \frac{1}{\sqrt{2n}}\sum_{j=1}^{n} P_j, \tag{55}$$

which obey the relations

$$[X_0, P_0] = \tfrac{1}{2}\mathrm{i} \tag{56}$$

and

$$\langle(\Delta X_0)^2\rangle\langle(\Delta P_0)^2\rangle \geqslant \tfrac{1}{16}, \tag{57}$$

where $\langle(\Delta X_0)^2\rangle \equiv \langle X_0^2\rangle - \langle X_0\rangle^2$. Due to equations (47) and (48), we know that the expectation values of X_0 and P_0 in $|r\rangle$, $\langle r|X_0|r\rangle = \langle r|P_0|r\rangle = 0$, so their variances are

$$\begin{aligned}\langle(\Delta X_0)^2\rangle = \langle r|X_0^2|r\rangle &= \langle\vec{0}|U_n^{-1}(r)X_0U_n(r)U_n^{-1}(r)X_0U_n(r)|\vec{0}\rangle \\ &= \frac{1}{2n}\sum_{j=1}^{n}(\mathrm{e}^{rG})_{kj}\sum_{i=1}^{n}(\mathrm{e}^{rG})_{li}\langle\vec{0}|X_kX_l|\vec{0}\rangle \\ &= \frac{1}{4n}\sum_{i,j=1}^{n}(\mathrm{e}^{2rG})_{ij}\end{aligned} \tag{58}$$

and

$$\begin{aligned}\langle(\Delta P_0)^2\rangle = \langle r|P_0^2|r\rangle &= \frac{1}{2n}\sum_{j=1}^{n}(\mathrm{e}^{-rG})_{kj}\sum_{i=1}^{n}(\mathrm{e}^{-rG})_{li}\langle\vec{0}|P_kP_l|\vec{0}\rangle \\ &= \frac{1}{4n}\sum_{i,j=1}^{n}(\mathrm{e}^{-2rG})_{ij}.\end{aligned} \tag{59}$$

According to equation (40),

$$\begin{aligned}\sum_{j,k=1}^{n}(\mathrm{e}^{2rG})_{jk} &= \sum_{l=0}^{\infty}\frac{(2r)^l}{l!}\sum_{j,k=1}^{n}(G^l)_{jk} \\ &= n\cosh 2r + \left(\frac{2}{\lambda} - n\right)\sinh 2r \\ &= \frac{1}{\lambda}\,\mathrm{e}^{2r} + \left(n - \frac{1}{\lambda}\right)\mathrm{e}^{-2r},\end{aligned} \tag{60}$$

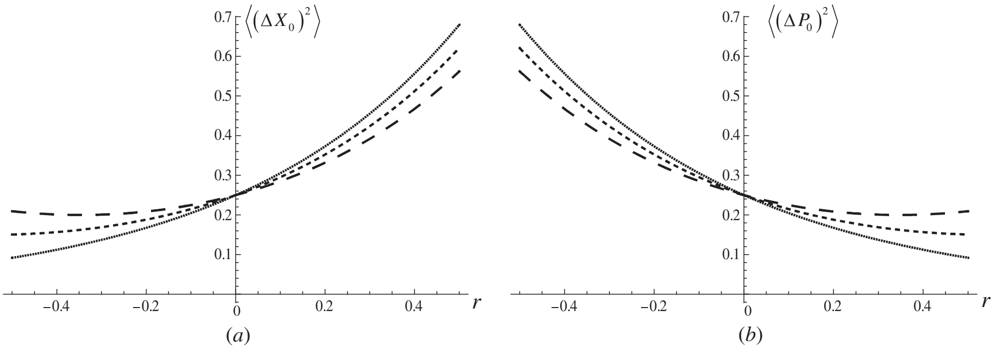

Figure 1. Variances of quadratures in the two-mode state $|r\rangle$ ($n = 2$) as a function of r for different parameters ε_i and λ as follows: $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}, \lambda = \frac{1}{2}$ (solid line); $\varepsilon_1 = \frac{1}{3}, \varepsilon_2 = \frac{2}{3}, \lambda = \frac{5}{9}$ (dashed line) and $\varepsilon_1 = \frac{1}{4}, \varepsilon_2 = \frac{3}{4}, \lambda = \frac{5}{8}$ (dotted line).

so

$$\langle(\Delta X_0)^2\rangle = \frac{1}{4n\lambda}\,e^{2r} + \left(\frac{1}{4} - \frac{1}{4n\lambda}\right)e^{-2r}, \tag{61}$$

and

$$\langle(\Delta P_0)^2\rangle = \frac{1}{4n\lambda}\,e^{-2r} + \left(\frac{1}{4} - \frac{1}{4n\lambda}\right)e^{2r}. \tag{62}$$

Because $\lambda \equiv \sum_{j=1}^{n}\varepsilon_j^2$, $\varepsilon_j = m_j/M < 1$, $\sum_{j=1}^{n}\varepsilon_j = 1$, according to the well-known inequality $\sqrt{\frac{\sum_{j=0}^{n}\varepsilon_j^2}{n}} \geqslant \frac{\sum_{j=0}^{n}\varepsilon_j}{n}$, i.e. $\frac{1}{n} \leqslant \lambda < 1$, we have

$$\begin{aligned}\langle(\Delta X_0)^2\rangle\langle(\Delta P_0)^2\rangle &= \frac{1}{16} - \frac{1}{8n\lambda} + \frac{1}{8n^2\lambda^2} + \left(\frac{1}{8n\lambda} - \frac{1}{8n^2\lambda^2}\right)\cosh 4r \\ &\geqslant \frac{1}{16},\end{aligned} \tag{63}$$

which agrees with equation (57). Especially, when $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = \frac{1}{n} \to \lambda = \frac{1}{n}$, both equations (61) and (62) recover as the standard form

$$\langle(\Delta X_0)^2\rangle = \tfrac{1}{4}\,e^{2r}, \qquad \langle(\Delta P_0)^2\rangle = \tfrac{1}{4}\,e^{-2r}. \tag{64}$$

In figures 1 and 2, we plot the graph of the variances $\langle(\Delta X_0)^2\rangle$ and $\langle(\Delta P_0)^2\rangle$ in the state $|r\rangle$ for the $n = 2$ and $n = 3$ cases, respectively. It is shown from figures 1 and 2 that the variances are sensitive to the different parameters ε_i and λ and for $r > 0$, $\langle(\Delta X_0)^2\rangle$ is always more than $\frac{1}{4}$ and $\langle(\Delta P_0)^2\rangle$ is always less than $\frac{1}{4}$, while for $r < 0$ the case is quite the contrary. Thus, for the generalized states $|r\rangle$, there exists a squeezing effect.

The concept of higher-order squeezing [19] is another aspect for revealing nonclassical characters of a quantum state. When the $2m$-th moment in a state is less than that in the vacuum state, this state is said to be squeezed to order $2m$. Using equation (55), we now calculate the $2m$-th moment of the n-mode quadrature X_0 in the state $|r\rangle$,

$$\begin{aligned}\langle(\Delta X_0)^{2m}\rangle &= \langle r|(\Delta X_0)^{2m}|r\rangle = \langle\vec{0}|U_n^{-1}(r)(\Delta X_0)^{2m}U_n(r)|\vec{0}\rangle \\ &= \left(\frac{1}{4n}\right)^m\langle\vec{0}|\left\{\Delta\sum_{j=1}^{n}\left[(e^{rG})_{jk}\left(a_k + a_k^\dagger\right)\right]\right\}^{2m}|\vec{0}\rangle.\end{aligned} \tag{65}$$

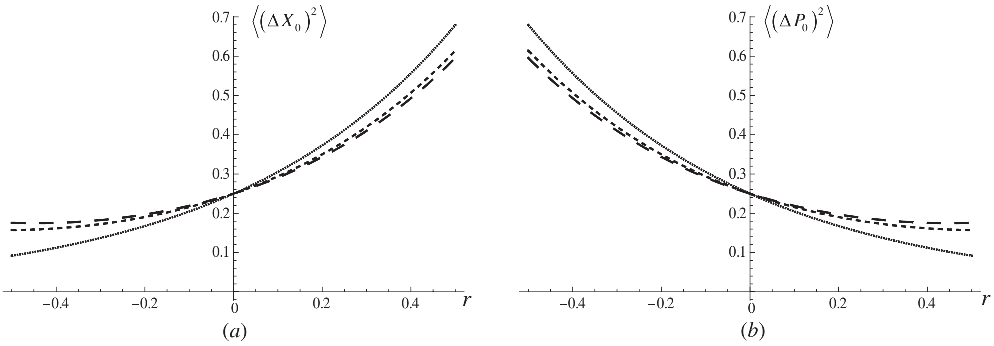

Figure 2. Variances of quadratures in the three-mode state $|r\rangle$ ($n = 3$) as a function of r for different parameters ε_i and λ as follows: $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}$, $\lambda = \frac{1}{3}$(solid line); $\varepsilon_1 = \varepsilon_2 = \frac{1}{4}$, $\varepsilon_3 = \frac{1}{2}$, $\lambda = \frac{3}{8}$(dashed line) and $\varepsilon_1 = \frac{1}{6}$, $\varepsilon_2 = \frac{1}{3}$, $\varepsilon_3 = \frac{1}{2}$, $\lambda = \frac{7}{18}$(dotted line).

Note that the formula [20]

$$(\Delta F)^{2m} = \sum_{k=0}^{m} \frac{(2m)!}{(2m-2k)!k!} : (\Delta F)^{2m-2k} : \left(\frac{\sum_{j=1}^{n} \xi_j \zeta_j}{2}\right)^k, \tag{66}$$

where $\Delta F \equiv F - \langle F\rangle$ with $F = \sum_{j=1}^{n} \left(\xi_j a_j + \zeta_j a_j^\dagger\right)$. Letting $Q = \sum_{j=1}^{n} \left[(\mathrm{e}^{rG})_{jk}\left(a_k + a_k^\dagger\right)\right]$ and using equation (60), we obtain

$$(\Delta Q)^{2m} = \sum_{k=0}^{m} \frac{(2m)!}{(2m-2k)!k!} : (\Delta Q)^{2m-2k} : \left(\frac{\frac{1}{\lambda}\,\mathrm{e}^{2r} + \left(n - \frac{1}{\lambda}\right)\mathrm{e}^{-2r}}{2}\right)^k. \tag{67}$$

By substituting equation (67) into equation (65), and using $\langle\vec{0}|(\Delta Q)^{2m-2k}|\vec{0}\rangle = \delta_{mk}$, equation (67) becomes

$$\langle(\Delta X_0)^{2m}\rangle = \left(\frac{1}{4}\right)^m (2m-1)!!\left(\frac{1}{n\lambda}\,\mathrm{e}^{2r} + \left(1 - \frac{1}{n\lambda}\right)\mathrm{e}^{-2r}\right)^m, \tag{68}$$

where $(2m-1)!! = 1\cdot 3\cdot 5\cdots(2m-1)$. Similarly, we have

$$\langle(\Delta P_0)^{2m}\rangle = \left(\frac{1}{4}\right)^m (2m-1)!!\left(\frac{1}{n\lambda}\,\mathrm{e}^{-2r} + \left(1 - \frac{1}{n\lambda}\right)\mathrm{e}^{2r}\right)^m. \tag{69}$$

It is clear that when $m = 1$, equations (68) and (69) reduce to equations (61) and (62), respectively. In order to see clearly the variations of the $2m$-th moment of quadratures in $|r\rangle$, the figures are plotted in figure 3 for several different m values, respectively, where we only consider the three-mode case, namely, $n = 3$. From figure 3, for the 4th moment of quadratures, when $r > 0$ ($r < 0$), $(\Delta X_0)^4 > \frac{3}{16}$ $\left((\Delta X_0)^4 < \frac{3}{16}\right)$ and $(\Delta P_0)^4 < \frac{3}{16}$ $\left((\Delta P_0)^4 > \frac{3}{16}\right)$, which implies that $|r\rangle$ is squeezed to ordered 4. In fact, for the other case, the results are similar, so $|r\rangle$ is squeezed not merely to the second order but to all higher orders.

5. Violation of the Bell inequality for the state $|r\rangle$

In this section, we examine the violation of the Bell inequality for the state $|r\rangle$ by using the formalism of the Wigner representation in phase space based on the parity operator and

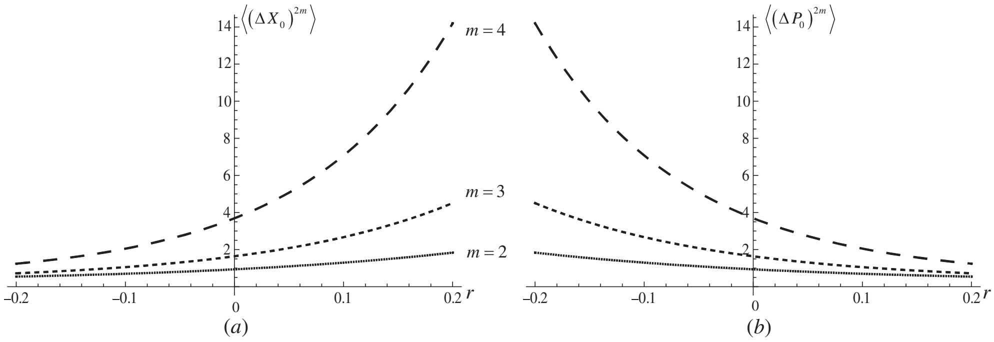

Figure 3. $2m$-th moment of quadratures in the three-mode state $|r\rangle$ ($n = 3$) as a function of r for several different $m = 2, 3, 4$ values, respectively.

the displacement operation. Wigner function representation of the Bell inequality has been developed using a parity operator as a quantum observable [21–23].

For the multi-mode system, the correlation function is the expectation of the operator

$$\Pi(\vec{\alpha}) = \bigotimes_{j=1}^{n} \Pi_j(\alpha_j) = \bigotimes_{j=1}^{n} D_j(\alpha_j)(-1)^{N_j} D_j^{\dagger}(\alpha_j), \tag{70}$$

which is an equivalent definition of the Wigner operator [21–24]; the corresponding Wigner function is

$$W(\vec{x}, \vec{p}) = W(\vec{\alpha}) = \frac{1}{\pi^n}\langle \Pi(\vec{\alpha})\rangle, \tag{71}$$

where $\vec{x} \equiv (x_1, x_2, \ldots, x_n)$, $\vec{p} \equiv (p_1, p_2, \ldots, p_n)$, $\vec{\alpha} = \frac{1}{\sqrt{2}}(\vec{x} + \mathrm{i}\vec{p}) \equiv (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $D_j(\alpha_j) = \exp\left(\alpha_j a_j^{\dagger} - \alpha_j^{*} a_j\right)$ is a displacement operator. $(-1)^{N_j}$ corresponds to the measurement of an even (+1) or an odd (−1) number of photons in mode j. Within the framework of local realistic theories, the Wigner representation of the Bell inequality is of the form [25–27]

$$|B(n)| \leqslant 2, \tag{72}$$

where $B(n)$ is a combination of $W(\vec{\alpha})$. For example, for the two-mode case, i.e. $n = 2$, the Bell inequality is expressed as

$$B(2) = \pi^2[W(\alpha_1, \alpha_2) + W(\alpha_1', \alpha_2) + W(\alpha_1, \alpha_2') - W(\alpha_1', \alpha_2')], \tag{73}$$

while for $n = 3$,

$$B(3) = \pi^3[W(\alpha_1, \alpha_2, \alpha_3') + W(\alpha_1, \alpha_2', \alpha_3) + W(\alpha_1', \alpha_2, \alpha_3) - W(\alpha_1', \alpha_2', \alpha_3')]. \tag{74}$$

In order to examine the violation of the Bell inequality for the state $|r\rangle$, we first calculate the Wigner function of the state $|r\rangle$. Remember that the Weyl ordered form of the n-mode Wigner operator is expressed as [28]

$$\Delta(\vec{x}, \vec{p}) = \bigotimes_{j=1}^{n} \Delta_j(x_j, p_j) = \begin{matrix}\cdot\\ \cdot\\ \cdot\end{matrix}\ \delta(\vec{x} - \vec{X})\delta(\vec{p} - \vec{P})\ \begin{matrix}\cdot\\ \cdot\\ \cdot\end{matrix} \tag{75}$$

where $\vec{X} \equiv (X_1, X_2, \ldots, X_n)$, $\vec{P} \equiv (P_1, P_2, \ldots, P_n)$ and $\begin{matrix}\cdot\\ \cdot\\ \cdot\end{matrix}\ \begin{matrix}\cdot\\ \cdot\\ \cdot\end{matrix}$ denotes the Weyl ordering. The normally ordered form of $\Delta(\vec{x}, \vec{p})$ is [24]

$$\Delta(\vec{x}, \vec{p}) = \frac{1}{\pi^n} : \exp[-(\vec{x} - \vec{X})^2 - (\vec{p} - \vec{P})^2] : . \tag{76}$$

Then according to the Weyl ordering invariance under similar transformations [29] and the Weyl quantization rule, using equations (47) and (48), we have

$$U_n^{-1}(r)\Delta(\vec{x},\vec{p})U_n(r) = \vdots\, \delta[\vec{x} - U_n^{-1}(r)\vec{X}U_n(r)]\delta[\vec{p} - U_n^{-1}(r)\vec{P}U_n(r)] \,\vdots$$
$$= \vdots\, \delta(\vec{x} - \mathrm{e}^{rG}\vec{X})\delta(\vec{p} - \mathrm{e}^{-rG}\vec{P}) \,\vdots$$
$$= \vdots\, \delta(\mathrm{e}^{-rG}\vec{x} - \vec{X})\delta(\mathrm{e}^{rG}\vec{p} - \vec{P}) \,\vdots$$
$$= \Delta(\mathrm{e}^{-rG}\vec{x}, \mathrm{e}^{rG}\vec{p}). \tag{77}$$

Thus using equations (76) and (77), the Wigner function of $|r\rangle$ is

$$W(\vec{x},\vec{p}) = \langle\vec{0}|U_n^{-1}(r)\Delta(\vec{x},\vec{p})U_n(r)|\vec{0}\rangle$$
$$= \frac{1}{\pi^n}\langle\vec{0}| : \exp[-(\mathrm{e}^{-rG}\vec{x} - \vec{X})^2 - (\mathrm{e}^{rG}\vec{p} - \vec{P})^2] : |\vec{0}\rangle$$
$$= \frac{1}{\pi^n}\exp[-x_j(\mathrm{e}^{-2rG})_{jk}x_k - p_j(\mathrm{e}^{2rG})_{jk}p_k]. \tag{78}$$

Considering equations (39) and (41) and $x_j = \frac{1}{\sqrt{2}}(\alpha_j + \alpha_j^*)$, $p_j = \frac{1}{i\sqrt{2}}(\alpha_j - \alpha_j^*)$, equation (78) can be rewritten as

$$W(\vec{x},\vec{p}) = \frac{1}{\pi^n}\exp\left[-\sum_{j=1}^{n}\left(x_j^2 + p_j^2\right)\cosh 2r + \sum_{j=1}^{n}\left(\frac{2\varepsilon_j^2}{\lambda} - 1\right)\left(x_j^2 - p_j^2\right)\sinh 2r \right.$$
$$\left. + \frac{4}{\lambda}\sum_{j>k=1}^{n}\varepsilon_j\varepsilon_k(x_jx_k - p_jp_k)\sinh 2r\right]$$
$$= \frac{1}{\pi^n}\exp\left[-2\sum_{j=1}^{n}|\alpha_j|^2\cosh 2r + \sum_{j=1}^{n}\left(\frac{2\varepsilon_j^2}{\lambda} - 1\right)\left(\alpha_j^2 + \alpha_j^{*2}\right)\sinh 2r \right.$$
$$\left. + \frac{4}{\lambda}\sum_{j>k=1}^{n}\varepsilon_j\varepsilon_k\left(\alpha_j\alpha_k + \alpha_j^*\alpha_k^*\right)\sinh 2r\right] \equiv W(\vec{\alpha}). \tag{79}$$

Especially, when $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = \frac{1}{n} \to \lambda = \frac{1}{n}$,

$$W_n(\vec{\alpha}) = \frac{1}{\pi^n}\exp\left[-2\sum_{j=1}^{n}|\alpha_j|^2\cosh 2r + \frac{4}{n}\sum_{j>k=1}^{n}\left(\alpha_j\alpha_k + \alpha_j^*\alpha_k^*\right)\sinh 2r \right.$$
$$\left. + \left(\frac{2}{n} - 1\right)\sum_{j=1}^{n}\left(\alpha_j^2 + \alpha_j^{*2}\right)\sinh 2r\right], \tag{80}$$

which is the same as the result of equation (16) given in [12].

For the $n = 2$ case, we examine the violation of the Bell inequality where $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$, $\alpha_1 = \alpha_2 = 0$ and $\alpha_1' = -\alpha_2' = b$ (b is a positive constant associated with the displacement magnitude). From these quantities, considering equations (73) and (80) and in the limit of large r ($\cosh 2r \simeq \mathrm{e}^{-2r}/2$) as well as small b, we obtain the following equation:

$$B(2) = \pi^2[W(0,0) + W(b,0) + W(0,-b) - W(b,-b)]$$
$$= 1 + 2\exp(-2b^2\cosh 2r) - \exp(-4b^2\,\mathrm{e}^{2r})$$
$$\simeq 1 + 2\exp(-b^2\,\mathrm{e}^{2r}) - \exp(-4b^2\,\mathrm{e}^{2r}), \tag{81}$$

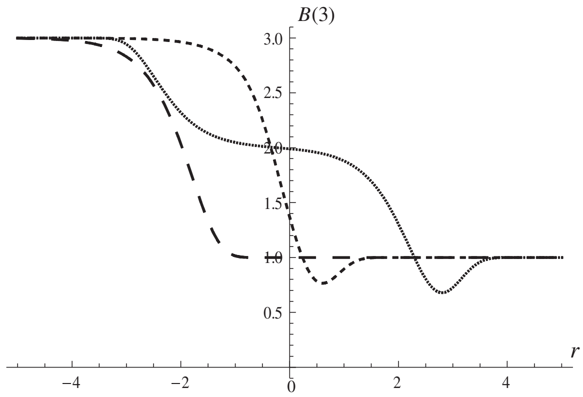

Figure 4. Violation of the Bell inequality $B(3)$ for the three-mode state $|r\rangle$ ($n = 3$) by using parity measurements with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}$, $\alpha_1 = \alpha_2 = \alpha_3' = 0$, $\alpha_3 = -b$ and $\alpha_1' = \alpha_2' = b$, where $b = 0.05$ (solid line), $b = 0.5$ (dashed line) and $b = 3$ (dotted line).

which is maximized for $b^2\,\mathrm{e}^{2r} = \frac{1}{3}\ln 2$: $B(2)_{\max} = 2.19$. This is a clear violation of the inequality $|B(2)| \leqslant 2$.

Taking $n = 3$ as another example, from equation (74) the Bell inequality equation has 12 variables, and it is highly nontrivial to find the global maximum values of $B(3)$ for all 12 variables. Fortunately, some local maximum values which violate the Bell inequality can be found numerically using the method of steepest descent. For this purpose, in figure 4 we plot the maximal Bell violation as a function of r, where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}$, $\alpha_1 = \alpha_2 = \alpha_3' = 0$, $\alpha_3 = -b$ and $\alpha_1' = \alpha_2' = b$. It is clear from figure 4 that the maximal-Bell violation increases with the squeezing parameter $|r|$ with $r < 0$ and the maximal value of $B(3)$ reaches 3, which implies that the Bell inequality for the three-mode case is violated. These results can be confirmed analytically. In fact, we put the above-chosen combinations into equation (74). For large $|r|$ with $r < 0$, $\cosh 2r \to \mathrm{e}^{-2r}/2$ and $\sinh 2r \to -\mathrm{e}^{-2r}/2$. After some calculation, $B(3)$ equals $3 - \exp\left(-\frac{8}{3}\,\mathrm{e}^{-2r}b^2\right)$. When $\mathrm{e}^{-2r}b^2$ is large enough, $B(3) \to 3$.

6. Conclusions

In summary, by constructing the generalized multi-partite entangled state representation and introducing the ket-bra integral in this representation, we find a new set of generalized bosonic realization of the $SU(1, 1)$ algebra, which can compose the generalized multi-mode squeezing operator $U_n(r)$. The intrinsic relation between the multi-mode entangled states of continuum variables and squeezing operator is clearly shown. The explicit multi-mode squeezed vacuum state $|r\rangle$ is obtained; the variances of the n-mode quadratures and higher-order squeezing for $|r\rangle$ are examined; the violation of the Bell inequality of $|r\rangle$ is examined. Finally, we mention that the new squeezing operator may be applied to the quantum theory of multi-photon absorption and emission in nonlinear optical processes, or tackling Bose–Einstein condensation in dilute gases, or deriving the ground state of trapped bosons, or obtaining collective modes in nuclear physics.

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